CARTAN INVARIANTS FOR THE RESTRICTED TORAL RANK TWO CONTACT LIE ALGEBRA

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Abstract. Restricted modules for the restricted toral rank two contact Lie algebra are considered. Contragredients of the simple modules, Cartan invariants, and dimensions of the simple modules and their projective covers are determined.

Let L be a finite dimensional restricted Lie algebra. All L-modules in this paper are assumed to be left, restricted and finite dimensional over the defining field. Each simple L-module has a projective cover; the multiplicities of the composition factors of the various projective covers are called *Cartan invariants*.

Here, we use the method of [3] to compute the Cartan invariants for the restricted toral rank two contact Lie algebra $K(3,\underline{\underline{1}})$. To carry out the computation, one needs to know the simple modules and their multiplicities as composition factors of certain induced modules. In [4]–which considered restricted contact Lie algebras of arbitrary toral rank–it was shown that these multiplicities are generically one, that is, the induced modules are, with a few exceptions, simple (see 1.1 below). Although it is not known at the time of this writing, it is expected that (for arbitrary toral rank) the few exceptional induced modules will not be simple. At least this is the case for the algebra $K(3,\underline{\underline{1}})$ as will be shown in this paper (see 6.1).

In addition to the Cartan invariants for $K(3,\underline{\underline{1}})$ we will compute the dimensions of the simple modules which will give in turn the dimensions of their projective covers. Also, we determine the contragredient of each simple module.

I thank the referee for the improvement in 2.3(2) and its proof as well as for other useful comments.

1. Statement of Main Results

Let F be an algebraically closed field of characteristic $p > 2$ and let $n = 2r + 1$ with $r \in \mathbb{N}$. For $1 \leq k \leq n$ let ε_k be the n-tuple with jth component δ_{jk} (Kronecker delta). Set $A = \{a = \sum_{k=1}^{n} a_k\varepsilon_k \,|\, 0 \leq a_k < p\} \subset \mathbb{Z}^n$. For $a, b \in A$, define $\binom{a}{b} := \prod_k \binom{a_k}{b_k}$. The factors on the right are the usual binomial coefficients with the convention that $\binom{i}{j} = 0$ unless $0 \leq j \leq i$. The vector space $\mathfrak{A}$ with F-basis $\{x^{(a)} \,|\, a \in A\}$ becomes an associative F-algebra by defining

$$x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)}$$

1991 *Mathematics Subject Classification.* Primary 17B50; Secondary 17B10.

(where $x^{(c)} = 0$ if $c \notin A$) and by extending this product linearly to $\mathfrak{A}$. $\mathfrak{A}$ is called the *divided power algebra.* It is a graded algebra with ith homogeneous component $\mathfrak{A}_i$ defined to be the F-span of $\{x^{(a)} \mid |a| + a_n = i\}$ where $|a| := \sum_{k=1}^{n} a_k$.

If $n+3 \equiv 0 \pmod p$ let K be the F-span of $\{x^{(a)} \mid a \neq \sum_k (p-1)\varepsilon_k\}$ and otherwise let $K = \mathfrak{A}$. Section 3 describes a way to give K the structure of a restricted Lie algebra. With this structure, K (denoted $K(3, \underline{\underline{1}})$ in [9]) is called the (*toral rank* $r+1$) *restricted contact Lie algebra.* The toral rank refers to the dimension of a maximal torus of K. Also, K has a restricted grading with ith homogeneous component K_i defined to be the F-span of $\{x^{(a)} \mid \|a\| = i\}$ where $\|a\| = |a| + a_n - 2$. Note that $K_i \subseteq \mathfrak{A}_{i+2}$.

For $1 \leq k \leq 2r$, set

$$\sigma(k) = \begin{cases} 1 & 1 \leq k \leq r, \\ -1 & r < k \leq 2r, \end{cases}$$

and $k' = k + \sigma(k)r$.

Let $\Lambda = \{\lambda = \sum_{i=1}^{r+1} \lambda_i \varepsilon_i \mid \lambda_i \in \mathbb{F}_p\} = (\mathbb{F}_p)^{r+1}$ (viewing ε_i as an $r+1$-tuple). For a K_0-module V and $\lambda \in \Lambda$ define $V_\lambda = \{v \in V \mid x^{(\varepsilon_i + \varepsilon_{i'})} v = \lambda_i v \ (1 \leq i \leq r) \text{ and } x^{(\varepsilon_n)} v = \lambda_{r+1} v\}$. Any vector in V_λ is said to have *weight* λ. A nonzero vector $m \in V_\lambda$ is a *maximal vector (of weight* λ) if $x^{(\varepsilon_i + \varepsilon_j)} m = 0$ for all $(i,j) \in I := \{(i,j) \mid 1 \leq i, j \leq r \text{ or } 1 \leq i \leq r, i' < j \leq 2r\}$.

For each $\lambda \in \Lambda$ there exists a simple (restricted) K_0-module $L_0(\lambda)$ possessing a unique (up to scalar multiple) maximal vector of weight λ. Moreover, $\{L_0(\lambda) \mid \lambda \in \Lambda\}$ is a complete set of representatives for the isomorphism classes of simple K_0-modules. In fact, K_0 is the direct sum of its p-ideals $\sum_{1 \leq i,j \leq 2r} F x^{(\varepsilon_i + \varepsilon_j)} \cong \mathfrak{sp}(2r)$ and $F x^{(\varepsilon_n)} \cong F$ and it is easy to see that $L_0(\lambda)$ is a simple $\mathfrak{sp}(2r)$-module on which $x^{(\varepsilon_n)}$ acts as multiplication by λ_{r+1}, so classical theory applies.

If L is a restricted Lie algebra, its restricted universal enveloping algebra (u-algebra) is denoted $u(L)$.

Set $N^+ = \sum_{i>0} K_i$. Then $N^+ \lhd N^+ + K_0 =: K^+$ and $K^+/N^+ \cong K_0$. In particular, any K_0-module becomes a K^+-module in the natural way.

In [2] it was shown that for $\lambda \in \Lambda$, the K-module $Z(\lambda) = u(K) \otimes_{u(K^+)} L_0(\lambda)$ possesses a unique maximal submodule which is graded with respect to the natural grading of $Z(\lambda)$ (cf. paragraph before 3.3 below). The quotient $L(\lambda)$ of $Z(\lambda)$ by this maximal submodule is simple and graded with homogeneous component of greatest degree K_0-isomorphic to $L_0(\lambda)$. $\{L(\lambda) \mid \lambda \in \Lambda\}$ is a complete set of representatives for the isomorphism classes of simple K-modules. Moreover, denoting by $L(\lambda)[i]$ the ith suspension of $L(\lambda)$ (so that $L(\lambda)[i]$ is the graded K-module with jth homogeneous component $L(\lambda)_{i+j}$), $\{L(\lambda)[i] \mid \lambda \in \Lambda, i \in \mathbb{Z}\}$ is a complete set of representatives for the isomorphism classes of simple graded K-modules. Consequently, if V is a simple graded K-module with homogeneous component of highest degree K_0-isomorphic to $L_0(\lambda)$, then $V \cong L(\lambda)$ as K-modules.

For $1 \leq k \leq r+1$ set $\zeta_k = -\sum_{i=1}^{r-k+1} \varepsilon_i$ (the empty sum being zero). A weight $\lambda \in \Lambda$ is *exceptional* if $\lambda = \zeta_k + (\pm k - r - 1)\varepsilon_{r+1}$ for some k $(1 \leq k \leq r+1)$. The following theorem was proved in [4].

1.1 Theorem. *If $\lambda \in \Lambda$ is not exceptional, then $L(\lambda) \cong Z(\lambda)$.*

In order to state the main results of this paper, we assume $n = 3$ so that K is the toral rank two restricted contact Lie algebra. Note that in this case, the exceptional weights are $(0,0), (0,-4), (-1,-1)$ and $(-1,-3)$.

In the following theorem V^* denotes the contragredient of the K-module V.

Theorem (see 5.3)**.**

(1) *If $\lambda \in \Lambda$ is not exceptional, then $L(\lambda)^* \cong L(\lambda_1, -\lambda_2 - 4)$.*
(2) $L(0,0)^* \cong L(0,0)$.

(3) $L(0,-4)^* \cong L(-1,-1)$.
(4) $L(-1,-3)^* \cong L(-1,-3)$.

If $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, then view $\lambda_i \in \mathbb{Z}$ with $-p < \lambda_i \le 0$ by identifying $\mathbb{F}_p$ with $\mathbb{Z}/p\mathbb{Z}$ and using the coset representative in the indicated range.

Theorem (see 5.5)**.**
(1) *If* $\lambda \in \Lambda$ *is not exceptional, then* $\dim_F L(\lambda) = p^3(1-\lambda_1)$.
(2) $\dim_F L(0,0) = 1$.
(3) $\dim_F L(0,-4) = \dim_F L(-1,-1) = p^3 - 1$.
(4) $\dim_F L(-1,-3) = p^3 - 2$.

Let $P(\lambda)$ denote the projective cover of $L(\lambda)$ and let $C_{\lambda\mu}$ be the multiplicity of $L(\mu)$ as a composition factor of $P(\lambda)$. $C_{\lambda\mu}$ is called a *Cartan invariant* and the matrix $C = [C_{\lambda\mu}]$ is called the *Cartan matrix* of K. (All indexing by Λ is assumed to be relative to a fixed ordering.)

Theorem (see 6.3 and 6.4)**.** *Assume* $p > 3$. $C = p^\beta X\, {}^tX$ *where* $\beta = p^3 - 12$, $X = [X_\lambda]$ *is the* $p^2 \times 1$*-matrix with*

$$X_\lambda = \begin{cases} 16 & \lambda = (0,0), \\ 4 & \lambda = (0,-4), (-1,-1) \text{ or } (-1,-3), \\ 1 & \lambda_1 = 1-p, \\ 2 & \text{otherwise}, \end{cases}$$

and ${}^tX$ *denotes the transpose of* X.

Theorem (see 6.5)**.** *Assume* $p > 3$. *For* $\lambda \in \Lambda$, $\dim_F P(\lambda) = p^\alpha X_\lambda$ *where* $\alpha = p^3 - 6$ *and* X_λ *is as above.*

2. Modules for the Witt algebra

Let n be arbitrary once again and for each k $(1 \le k \le n)$ let D_k be the derivation of $\mathfrak{A}$ given by $D_k x^{(a)} = x^{(a-\varepsilon_k)}$ with the convention that $x^{(b)} = 0$ if $b \notin A$. Then $W = \sum_k \mathfrak{A} D_k = \mathrm{Der}_F \mathfrak{A}$ is a restricted Lie algebra called the *Witt algebra*. It has as F-basis $\{x^{(a)} D_k \,|\, a \in A, 1 \le k \le n\}$. The bracket product in W satisfies

$$[x^{(a)} D_k, x^{(b)} D_l] = \binom{a+b-\varepsilon_k}{a} x^{(a+b-\varepsilon_k)} D_l - \binom{a+b-\varepsilon_l}{b} x^{(a+b-\varepsilon_l)} D_k.$$

View W as an $\mathfrak{A}$-module in the natural way and set $M = \mathrm{Hom}_{\mathfrak{A}}(W, \mathfrak{A})$. M is a free $\mathfrak{A}$-module with base $\{dx_1, \dots, dx_n\}$ where $x_k := x^{(\varepsilon_k)}$ and $d : \mathfrak{A} \to M$ is given by $dx : D \mapsto Dx$ $(x \in \mathfrak{A}, D \in W)$. M becomes a W-module by defining $D.\varphi = D \circ \varphi - \varphi \circ (\mathrm{ad}\, D)$ $(D \in W, \varphi \in M)$ (cf. [1, p. 125]).

The following lemma is easy to verify.

2.1 Lemma. $x^{(a)} D_i . dx_j = \begin{cases} \sum_k x^{(a-\varepsilon_k)} dx_k & i = j, \\ 0 & i \ne j. \end{cases}$

Let Ω denote the exterior algebra of M over $\mathfrak{A}$. For $1 \le k \le n$, set $\Gamma_k = \{\gamma = \sum_{i=1}^k \gamma_i \varepsilon_i \,|\, 1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_k \le n\}$ and for $\gamma \in \Gamma_k$ set $e_\gamma = dx_{\gamma_1} \wedge \cdots \wedge dx_{\gamma_k} \in \Omega$. Also, let $\Gamma_0 = \{\zeta\}$ where ζ denotes the empty tuple, $\zeta = (\ )$, and set $e_\zeta = 1 \in \mathfrak{A}$. Then $\Omega = \dot{\sum}_k \Omega_k$ where $\Omega_k := \sum_{\gamma \in \Gamma_k} \mathfrak{A} e_\gamma$ if $0 \le k \le n$ (and, for convenience, $\Omega_k = 0$ if $k < 0$ or $k > n$). For $0 \le k \le n$, Ω_k has F-basis $\{x^{(a)} e_\gamma \,|\, a \in A, \gamma \in \Gamma_k\}$.

The action of W on M extends uniquely to an action on Ω subject to the rules

$$D.(v\wedge w) = D.v\wedge w + v\wedge D.w$$
$$D.(xv) = (Dx)v + x(D.v)$$

for $D \in W$, $x \in \mathfrak{A}$ and $v, w \in \Omega$. It is clear from 2.1 that each Ω_k is a W-submodule of Ω.

Define an F-linear map $\delta_j : \Omega_j \to \Omega_{j+1}$ by setting

$$\delta_j(x^{(a)}e_\gamma) = \sum_{k=1}^{n} x^{(a-\varepsilon_k)}e_\gamma\wedge dx_k$$

if $0 \leq j \leq n$ and by setting $\delta_j = 0$ if $j < 0$ or $j > n$. It is straightforward to check that δ_j is a W-homomorphism and that $\delta_j\delta_{j-1} = 0$.

For $0 \leq j, k \leq n$ set

$$B_{jk} = \{x^{(a)}e_\gamma \in \Omega_j \mid k \notin \gamma, a_k \neq 0 \text{ and for } i < k, a_i = 0 \text{ if } i \notin \gamma \text{ and } a_i = p-1 \text{ if } i \in \gamma\}$$

where $k \in \gamma$ means $k = \gamma_i$ for some i. Let $B_j = \bigcup_k B_{jk}$.

2.2 Lemma. *For $0 \leq j < n$, $\delta_j(B_j)$ is an F-basis for* $\operatorname{im}\delta_j$.

Proof. Fix j with $0 \leq j < n$. It will first be shown that $\delta_j(B_j)$ spans $\operatorname{im}\delta_j$. Now $\operatorname{im}\delta_j$ is spanned by $\delta_j(\bigcup_k Y_k)$ where $Y_k = \{x^{(a)}e_\gamma \mid k \notin \gamma, a_k \neq 0\}$ since any standard basis element $x^{(a)}e_\gamma$ of Ω_j not in Y_k is in $\ker\delta_j$. Therefore, it is enough to prove that for each k, $\delta_j(Y_k)$ is contained in the F-span of $\delta_j(B_j)$, and this will be done by induction on k. Since $Y_1 = B_{j1}$, the first step is trivial. Now assume $k > 1$ and let $y = x^{(a)}e_\gamma$ be an arbitrary element of Y_k. If $a_i \neq 0$ for some $1 \leq i < k$ with $i \notin \gamma$, then $y \in Y_i$ and the induction hypothesis implies $\delta_j(y)$ is in the F-span of $\delta_j(B_j)$. So assume that $a_i = 0$ for each $1 \leq i < k$ with $i \notin \gamma$. Similarly, if $a_i = p-1$ for each $1 \leq i < k$ with $i \in \gamma$, then $y \in B_{jk}$ so that $\delta_j(y) \in \delta_j(B_j)$. So assume otherwise and let i be the least index for which $i \in \gamma$ and $a_i \neq p-1$. Define $\eta = (\gamma_1, \gamma_2, \ldots, \hat{i}, \ldots, \gamma_j)$ ("delete i") and $b = a + \varepsilon_i$. Then $\delta_{j-1}(x^{(b)}e_\eta) = \sum_l x^{(b-\varepsilon_l)}e_\eta\wedge dx_l = \pm y + \sum_{l>i} x^{(b)}e_\eta\wedge dx_l$. Each term in the sum on the right is either zero or contained in B_{ji}. Therefore, applying δ_j and using the fact that $\delta_j\delta_{j-1} = 0$ it follows that $\delta_j(y)$ is in the F-span of $\delta_j(B_j)$. This completes the proof that $\delta_j(B_j)$ spans $\operatorname{im}\delta_j$.

Next, let $1 \leq k \leq l \leq n$ and pick $y = x^{(a)}e_\gamma \in B_{jk}$ and $z = x^{(b)}e_\theta \in B_{jl}$. Assume $x^{(b-\varepsilon_i)}e_\theta\wedge dx_i = x^{(a-\varepsilon_k)}e_\gamma\wedge dx_k$ for some i. It will be shown that, under this assumption, $l = k$ and $z = y$. This, and the easily verified fact that no element of $\delta_j(B_j)$ is zero, will imply linear independence of $\delta_j(B_j)$. Since $x^{(a-\varepsilon_k)}e_\gamma\wedge dx_k \neq 0$, it follows that $i \geq l$. If $k \in \theta$, then $k < l$ which yields the contradiction $b - \varepsilon_i)_k = b_k = p-1 \neq a_k - 1 = (a-\varepsilon_k)_k$. Therefore, $k \notin \theta$. Since $e_\theta\wedge dx_i = e_\gamma\wedge dx_k \neq 0$ it must be the case that $i = k$. It now easily follows that $k = l$ and $z = y$, as desired. □

2.3 Corollary (cf. [8], Theorem 2.1)**.** *For $-1 \leq j < n$,*

(1) $\dim_F \operatorname{im}\delta_j = \binom{n-1}{j}(p^n - 1)$,

(2) $\dim_F \ker\delta_{j+1}/\operatorname{im}\delta_j = \binom{n}{j+1}$ *and W acts trivially on* $\ker\delta_{j+1}/\operatorname{im}\delta_j$.

Proof. Part (1) follows directly from 2.2 by a counting of the elements of B_j.

For (2), first note that $\dim_F \ker\delta_{j+1}/\operatorname{im}\delta_j = \dim_F\Omega_{j+1} - \dim_F\Omega_{j+1}/\ker\delta_{j+1} - \dim_F \operatorname{im}\delta_j = \binom{n}{j+1}p^n - \binom{n-1}{j+1}(p^n-1) - \binom{n-1}{j}(p^n-1) = \binom{n}{j+1}$ by part (1).

Therefore, all that remains to be proved is the second claim of (2). For $1 \leq k \leq n$, set $f_k = x^{((p-1)\varepsilon_k)}dx_k$ and for $\gamma \in \Gamma_{j+1}$, set $f_\gamma = f_{\gamma_1}\wedge f_{\gamma_2}\wedge\cdots\wedge f_{\gamma_{j+1}}$ ($f_\zeta = 1$). Then $f_\gamma \in \ker\delta_{j+1}$.

Moreover, $\{f_\gamma + \operatorname{im}\delta_j \mid \gamma \in \Gamma_{j+1}\}$ is clearly linearly independent and, since the cardinality of this set is $\binom{n}{j+1}$, it must be a basis for $\ker\delta_{j+1}/\operatorname{im}\delta_j$ by the previous paragraph. Let $x^{(a)}D_i \in W$. It suffices to show that $(x^{(a)}D_i).f_\gamma \in \operatorname{im}\delta_j$ for each $\gamma \in \Gamma_{j+1}$. If $j = -1$, then $f_\gamma = 1$ and so $(x^{(a)}D_i).f_\gamma = 0 \in \operatorname{im}\delta_j$. Therefore, assume $j > -1$. Now,

$$(x^{(a)}D_i).f_k = \begin{cases} \sum_l x^{(a+(p-1)\varepsilon_i-\varepsilon_l)}dx_l & \text{if } i = k \text{ and } a_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So $(x^{(a)}D_i).f_\gamma = 0$ unless $i \in \gamma$ and $a_i = 0$ in which case

$$(x^{(a)}D_i).f_\gamma = \pm\sum_l x^{(b-\varepsilon_l)}e_\eta \wedge dx_l = \delta_j(\pm x^{(b)}e_\eta) \in \operatorname{im}\delta_j,$$

where $b = a + \sum_k (p-1)\varepsilon_{\gamma_k}$ and $\eta = (\gamma_1, \gamma_2, \ldots, \hat{i}, \ldots, \gamma_{j+1})$ ("delete i"). This completes the proof. □

Let $w_K := dx_n + \sum_{k=1}^{2r}\sigma(k)x_k dx_{k'} \in \Omega_1$ and define $\tau_j : \Omega_j \to \Omega_{j+1}$ by $\tau_j(v) = w_K \wedge v$. Set $C_j = \{x^{(a)}e_\gamma \mid a \in A, \gamma \in \Gamma_j, n \notin \gamma\}$ $(0 \le j < n)$.

2.4 Lemma. *For $0 \le j < n$,*

(1) $\Omega_{j-1} \overset{\tau_{j-1}}{\to} \Omega_j \overset{\tau_j}{\to} \Omega_{j+1}$ *is exact,*
(2) $\tau_j(C_j)$ *is an F-basis for* $\operatorname{im}\tau_j$*,*
(3) $\dim_F \operatorname{im}\tau_j = \binom{n-1}{j}p^n$.

Proof. First observe that $\tau_j(C_j)$ is linearly independent, for if $v = x^{(a)}e_\gamma \in C_j$, then $\tau_j(v)$ is the sum of $x^{(a)}dx_n \wedge e_\gamma$ and scalar multiples of terms of the form $x^{(b)}e_\rho$ with $n \notin \rho$.

Next, since $w_K \wedge w_K = 0$, it follows that $\operatorname{im}\tau_{j-1} \subseteq \ker\tau_j$.

Therefore, $\binom{n-1}{j}p^n = |C_j| \le \dim_F \operatorname{im}\tau_j = \dim_F \Omega_j/\ker\tau_j \le \dim_F \Omega_j/\operatorname{im}\tau_{j-1} = \binom{n}{j}p^n - \binom{n-1}{j-1}p^n = \binom{n-1}{j}p^n$, the next to the last equality from the induction hypothesis (or the fact that $\operatorname{im}\tau_{-1} = \{0\}$ if $j = 0$). Hence, both inequalities are, in fact, equalities and all three statements are established. □

The element w_K defined above can be used to give an alternate definition of K (see [1]).

3. Modules for K (n arbitrary).

In section 1 an F-basis was given for the underlying vector space of the contact algebra K. Now the Witt algebra W will be used to introduce the bracket product on K. (For details, see [9].)

Define an F-linear map $D_K : K \to W$ by means of $D_K(f) = \sum_{k=1}^n f_k D_k$, where

$$f_k = x_k D_n(f) + \sigma(k')D_{k'}(f), \quad k \le 2r,$$

$$f_n = 2f - \sum_{k=1}^{2r}\sigma(k)x_k f_{k'}.$$

(These formulas are taken from [9]. They differ from those in [1] by some signs which explains why our definition of w_K above is different from that given in [1].) D_K is injective and its image is closed under both the bracket product and the p-mapping of W. Therefore, identifying K with its image under D_K, K is a restricted Lie algebra and $K < W$. In fact, K is a simple algebra.

To avoid confusion with the trivial bracket product on $\mathfrak{A}$, denote the induced bracket product on K by $\langle f, g\rangle$ $(f, g \in K)$. The following list of formulas will be useful.

3.1 Lemma ([9, p. 173]). (1) $\langle x^{(0)}, x^{(a)}\rangle = 2x^{(a-\varepsilon_n)}$.
(2) $\langle x^{(\varepsilon_k)}, x^{(a)}\rangle = \sigma(k)x^{(a-\varepsilon_{k'})} + (a_k+1)x^{(a+\varepsilon_k-\varepsilon_n)}, \quad 1 \le k \le 2r$.
(3) $\langle x^{(\varepsilon_n)}, x^{(a)}\rangle = \|a\|x^{(a)}$.
(4) $\langle x^{(\varepsilon_k+\varepsilon_{k'})}, x^{(a)}\rangle = \sigma(k)(a_{k'}-a_k)x^{(a)}, \quad 1 \le k \le 2r$.

For any nonzero finite dimensional graded vector space $V = \sum_i V_i$ define the *length* of V to be $M - m$ where M (resp., m) is the maximum (resp., minimum) of the set $\{i \mid V_i \neq 0\}$.

3.2 Lemma (cf. [7, Lemma 2.2]). *Let $V = \dot{\sum}_i V_i$ be a nonzero graded K-module and assume that the length of V is less than*

$$l := \begin{cases} (n+1)(p-1)-3 & p = 3, \\ (n+1)(p-1)-2 & p \neq 3. \end{cases}$$

Then K acts trivially on V.

Proof. The annihilator of V is an ideal of K which contains the nonzero component K_l. Since K is simple, the result follows. □

As in [4], set

$$T_k = \begin{cases} x_k & 1 \le k < n, \\ x^{(0)} & k = n, \end{cases}$$

and for $a = \sum_k a_k\varepsilon_k \in \mathbb{Z}^n$ define $T^a = \prod_{k=1}^n T_k^{a_k} \in u(K)$, where $T_k^i := 0$ if $i < 0$. It is a consequence of the p-mapping defined on K that $T_k^i = 0$ if $i \ge p$, so that $T^a = 0$ if and only if $a \notin A$.

The K-module $Z(\lambda) := u(K) \otimes_{u(K^+)} L_0(\lambda)$ is graded with ith homogeneous component $Z(\lambda)_i = \sum_{|a|+a_n=-i} T^a \otimes L_0(\lambda)$.

3.3 Lemma. *If N is a nonzero submodule of the K-module $Z(\lambda)$ ($\lambda \in \Lambda$), then $N \supseteq T^\omega \otimes L_0(\lambda)$ where $\omega = \sum_{k=1}^n (p-1)\varepsilon_k$.*

Proof. Let $0 \neq v \in N$. It follows from the PBW theorem that v can be written (uniquely) in the form

$$v = \sum_{a \in A} T^a \otimes s_a$$

with $s_a \in L_0(\lambda)$. Order A by setting $a < a'$ if for some k $(1 \le k \le n)$ $a_i = a'_i$ for all $i > k$ and $a_k < a'_k$. Let η be the least element for which $s_\eta \neq 0$ and set $y = \prod_{i=1}^n T_i^{p-1-\eta_i}$. Then, using 3.1, $T^\omega \otimes s_\eta = yv \in N$. Now $T^\omega \otimes L_0(\lambda)$ is a K_0-submodule of $Z(\lambda)$, namely, the homogeneous component of $Z(\lambda)$ of least degree. Moreover, $T^\omega \otimes L_0(\lambda)$ is simple so, since it intersects N nontrivially, it must be contained in N. □

For each $0 \le j \le n$, the K-module Ω_j is graded with ith homogeneous component being the F-span of all $x^{(a)}d\gamma$ ($a \in A, \gamma \in \Gamma_j$) for which $i = |a| + a_n + \begin{cases} j+1 & n \in \gamma, \\ j & n \notin \gamma. \end{cases}$

3.4 Lemma. *For each $0 \le j < n$, $\operatorname{im}\tau_j$ is a graded K-submodule of Ω_{j+1}.*

Proof. It is routine to check that for each $f \in K$, $D_K(f).w_K = 2D_n(f)w_K$. Therefore, if $v \in \Omega_j$ and $f \in K$, then

$$\begin{aligned} D_K(f).\tau_j(v) &= D_K(f).(w_K \wedge v) \\ &= (D_K(f).w_K)\wedge v + w_K \wedge (D_K(f).v) \\ &= w_K \wedge (2D_n(f)v + D_K(f).v) \in \operatorname{im}\tau_j. \end{aligned}$$

Since w_K is a homogeneous element of Ω_1 (of degree 2) it follows that the image under τ_j of a homogeneous element is homogeneous, so that $\operatorname{im}\tau_j$ is graded. $\square$

It should be pointed out that τ_j is *not* a K-homomorphism, in general.

4. Simple modules for K ($n = 3$).

For the rest of the paper, we assume $n = 3$ so that K is the toral rank two contact algebra.

It will be convenient to have explicit formulas for the actions of certain elements of K on the standard basis vectors of Ω.

4.1 Lemma. *Let $v = x^{(a)}e_\gamma \in \Omega$ and set* $\chi_k = \begin{cases} 1 & k \in \gamma, \\ 0 & k \notin \gamma. \end{cases}$

(1) $x^{(0)}.v = 2x^{(a-\varepsilon_3)}e_\gamma$.
(2) $x_1.v = x^{(a-\varepsilon_2)}e_\gamma$ *if* $a_3 = 0$ *and* $3 \notin \gamma$.
(3) $x_2.v = -x^{(a-\varepsilon_1)}e_\gamma$ *if* $a_3 = 0$ *and* $3 \notin \gamma$.
(4) $x^{(\varepsilon_1+\varepsilon_2)}.v = (a_2 - a_1 + \chi_2 - \chi_1)v$.
(5) $x_3.v = (a_1 + a_2 + 2a_3 + \chi_1 + \chi_2 + 2\chi_3)v$.
(6) $x^{(2\varepsilon_1)}.v = (a_1 + 1)x^{(a-\varepsilon_2+\varepsilon_1)}e_\gamma$ *if* $2 \notin \gamma$.

Proof. Recall that if $f \in K$, then $f.v = D_K(f).v$, by definition. The lemma now follows from 2.1, the action of W on Ω, and the following formulas: $D_K(x^{(0)}) = 2D_3$, $D_K(x_1) = D_2 + x_1D_3$, $D_K(x_2) = -D_1 + x_2D_3$, $D_K(x^{(\varepsilon_1+\varepsilon_2)}) = -x_1D_1 + x_2D_2$, $D_K(x_3) = x_1D_1 + x_2D_2 + 2x_3D_3$, $D_K(x^{(2\varepsilon_1)}) = x_1D_2$. $\square$

4.2 Lemma.

(1) $\Omega_0 \cong Z(0, -4)$.
(2) $\Omega_1/\operatorname{im}\tau_0 \cong Z(-1, -3)$.

Proof. As before, let $\omega = \sum_k (p-1)\varepsilon_k$.

(1) $x^{(\omega)}$ is a maximal vector of weight $(0, -4)$ (using 4.1(4,5,6)) in the one-dimensional space $(\Omega_0)_{4p-4}$ which is the homogeneous component of Ω_0 of greatest degree. Hence there is a K^+-isomorphism $L_0(0, -4) \to (\Omega_0)_{4p-4}$ (see 5.1 below) which induces a K-homomorphism $\varphi : Z(0, -4) \to \Omega_0$. Using 4.1(1,2,3), it can be seen that φ sends the space $T^\omega \otimes L_0(0, -4)$ onto $Fx^{(0)} \neq 0$ so that φ is injective, by 3.3. Since the dimension of each space is p^3, it follows that φ is an isomorphism.

(2) Here, there is a K^+-isomorphism $L_0(-1, -3) \to (\Omega_1/\operatorname{im}\tau_0)_{4p-3}$ since this last space has basis $\{x^{(\omega)}dx_1 + \operatorname{im}\tau_0, x^{(\omega)}dx_2 + \operatorname{im}\tau_0\}$ according to 2.4, and $x^{(\omega)}dx_1 + \operatorname{im}\tau_0$ is a maximal vector of weight $(-1, -3)$ (see also 5.1). This yields a K-homomorphism $\varphi : Z(-1, -3) \to \Omega_1/\operatorname{im}\tau_0$. Notice that φ sends $T^\omega \otimes L_0(-1, -3)$ onto the F-span of $\{x^{(0)}dx_1 + \operatorname{im}\tau_0, x^{(0)}dx_2 + \operatorname{im}\tau_0\}$ which is nonzero by 2.4. Hence 3.3 implies φ is injective. Since the dimension of each space is $2p^3$ (see 2.4 and 5.1), φ is an isomorphism. $\square$

Set $S_j = \operatorname{im}\tau_{j-1} + \ker\delta_j$ and $\bar{\Omega}_j = \Omega_j/S_j$ and let B_j be as in section 2. Let $E_1 = \{x^{(a)}dx_3 \mid a_1 \neq 0 \text{ or } a_2 \neq 0\}$ and $E_2 = \{x^{(a)}dx_2 \mid a = (1, 0, c) \text{ for some } c\}$ and set $E = E_1 \cup E_2$. Note that $E \subset B_1$. Define $H_0 = B_0$ and $H_1 = B_1 \backslash E$ and set $\bar{H}_j = \{h + S_j \mid h \in H_j\}$ $(j = 0, 1)$.

4.3 Lemma. *$\bar{H}_j$ is an F-basis for $\bar{\Omega}_j$ $(j = 0, 1)$.*

Proof. First, note that $S_0 = \ker\delta_0$, so that the case $j = 0$ is handled by 2.2.

Using 4.1, $v := \tau_0(x^{(\omega)}) = x^{(\omega)}dx_3$ (where $\omega = \sum_k (p-1)\varepsilon_k$) is a maximal vector of weight $(0, -2)$. Also Fv is the homogeneous component of $\operatorname{im}\tau_0$ of greatest degree. Hence, there is a K^+-isomorphism $L_0(0, -2) \to Fv$ which extends to a K-homomorphism $\varphi : Z(0, -2) \to \operatorname{im}\tau_0$.

Since $(0,-2)$ is not exceptional, $Z(0,-2)$ is simple (see 1.1) so that φ is an isomorphism. Hence, $\operatorname{im}\tau_0$ is simple and of dimension p^3. Also, $v \in \operatorname{im}\tau_0 \backslash \ker\delta_1$ so $\operatorname{im}\tau_0 \cap \ker\delta_1 = \{0\}$. This gives $\dim_F \bar{\Omega}_1 = \dim_F \Omega_1/\ker\delta_1 - \dim_F \operatorname{im}\tau_0 = |B_1| - p^3 = |\bar{H}_1|$. Therefore, it suffices to show $\bar{H}_1$ spans $\bar{\Omega}_1$ and for this it need only be shown that for each $b \in E$, $b + S_1$ is in the F-span of $\bar{H}_1$ (cf. 2.2).

Let $a \in A$ and consider the equation

$$(*) \qquad \tau_0(x^{(a)}) = x^{(a)}dx_3 - (a_2+1)x^{(a+\varepsilon_2)}dx_1 + (a_1+1)x^{(a+\varepsilon_1)}dx_2.$$

Assume $x^{(a)}dx_3 \in E_1$. If $a_1 = p-1$, then the equation implies that $x^{(a)}dx_3 \equiv (a_2+1)x^{(a+\varepsilon_2)}dx_1 \pmod{S_1}$ and since $x^{(a+\varepsilon_2)}dx_1$ is either zero or in H_1, this implies $x^{(a)}dx_3 + S_1$ is in the F-span of $\bar{H}_1$. The case $a_2 = p-1$ is handled similarly. Now assume $a_1 \neq p-1 \neq a_2$. Then

$$\delta_0(x^{(a+\varepsilon_1+\varepsilon_2)}) = x^{(a+\varepsilon_2)}dx_1 + x^{(a+\varepsilon_1)}dx_2 + x^{(a+\varepsilon_1+\varepsilon_2-\varepsilon_3)}dx_3.$$

Since $\delta_1\delta_0 = 0$, it follows that

$$x^{(a+\varepsilon_2)}dx_1 \equiv -x^{(a+\varepsilon_1)}dx_2 - x^{(a+\varepsilon_1+\varepsilon_2-\varepsilon_3)}dx_3 \pmod{\ker\delta_1},$$

whence, from (*),

$$x^{(a)}dx_3 \equiv -(a_1+a_2+2)x^{(a+\varepsilon_1)}dx_2 - (a_2+1)x^{(a+\varepsilon_1+\varepsilon_2-\varepsilon_3)}dx_3 \pmod{S_1}.$$

Since either $a_1 \neq 0$ or $a_2 \neq 0$ the first term on the right is in FH_1. By using reverse induction on a_1 it follows that $x^{(a)}dx_3 + S_1$ is in the F-span of $\bar{H}_1$.

Now assume $x^{(a)}dx_2 \in E_2$. The equations

$$\tau_0(x^{(a-\varepsilon_1)}) = x^{(a-\varepsilon_1)}dx_3 - x^{(a-\varepsilon_1+\varepsilon_2)}dx_1 + x^{(a)}dx_2$$

and

$$\delta_0(x^{(a+\varepsilon_2)}) = x^{(a+\varepsilon_2-\varepsilon_1)}dx_1 + x^{(a)}dx_2 + x^{(a+\varepsilon_2-\varepsilon_3)}dx_3$$

along with the fact that $x^{(a-\varepsilon_1)}dx_3 \in \ker\delta_1$ show that $2x^{(a)}dx_2 \equiv -x^{(a+\varepsilon_2-\varepsilon_3)}dx_3 \pmod{S_1}$. The previous paragraph now shows that $x^{(a)}dx_2 + S_1$ is in the F-span of $\bar{H}_1$ (recalling that $p \neq 2$). $\square$

4.4 Lemma. *$\bar{\Omega}_j\,(j = 0,1)$ has no submodule isomorphic to the one-dimensional trivial module F.*

Proof. $(j = 0)$ Let $v \in \bar{\Omega}_0$ be a vector on which K acts trivially. By 4.3 there are unique scalars $c_a \in F$ for which $v = \sum_{a\neq 0} c_a(x^{(a)} + S_0)$. First note that 4.1(5) implies $c_a = 0$ if $a = (0,0,1)$. Using 4.1(1), $0 = x^{(0)}.v = \sum_{a\neq 0} 2c_a(x^{(a-\varepsilon_3)} + S_0)$. For $a \neq 0$, either $c_a = 0$, $x^{(a-\varepsilon_3)} = 0$, or $x^{(a-\varepsilon_3)} + S_0$ is in $\bar{H}_0$. Moreover, the elements of $\bar{H}_0$ appearing are all distinct. Therefore, by linear independence, $c_a = 0$ if $a_3 \neq 0$. Next, 4.1(4), implies $c_a = 0$ if $a_1 \neq a_2$. So finally, using 4.1(2), $c_a = 0$ for all a.

$(j = 1)$ Proceeding as above, let $v \in \bar{\Omega}_1$ be a vector on which K acts trivially. Write v as a linear combination of the basis vectors in $\bar{H}_1$ (see 4.3): $v = \sum_{(a,\gamma)} c_{a,\gamma}(x^{(a)}e_\gamma + S_1)$ where $c_{a,\gamma} \in F$ and the sum is over all pairs (a,γ) for which $x^{(a)}e_\gamma \in H_1$. If $x^{(a)}e_\gamma \in B_{1,3}$ (definition before 2.2) and $a_3 = 1$, then 4.1(5) implies $x_3.x^{(a)}e_\gamma = 2x^{(a)}e_\gamma$ so that $c_{a,\gamma} = 0$. Now, using 4.1(1) as in the case $j = 0$ above, it follows that $c_{a,\gamma} = 0$ if $a_3 \neq 0$. Next, by using 4.1(4), $c_{a,\gamma} = 0$ if either $a = (p-1,1,0)$ and $\gamma = (1)$ or $a = (1,1,0)$ and $\gamma = (2)$. Therefore, 4.1(2) implies $c_{a,\gamma} = 0$ if $a_2 \neq 0$. Finally 4.1(4) implies each $c_{a,\gamma}$ is zero. $\square$

Since δ_j is a graded K-homomorphism, $\ker\delta_j$ is a graded K-submodule of Ω_j. Hence, by 3.4, S_j is a graded K-submodule of Ω_j. Therefore, $\bar{\Omega}_j$ is a graded K-module $(j = 0,1)$.

4.5 Theorem.

(1) $\bar{\Omega}_0 \cong L(0,-4)$.
(2) $\bar{\Omega}_1 \cong L(-1,-3)$.

Proof. (1) By 4.2, $\bar{\Omega}_0$ is a homomorphic image of $Z(0,-4)$, so it suffices to show that $\bar{\Omega}_0$ is simple. In fact, from the comments before 1.1, it is enough to show that $\bar{\Omega}_0$ is simple as a *graded* module. Suppose N is a simple proper graded submodule of $\bar{\Omega}_0$. Note that, by 4.3, the homogeneous component $(\bar{\Omega}_0)_i$ is zero if either $i < 1$ or $i > 4p-4$. Assume $N_{4p-4} \neq 0$. Then $N_{4p-4} = (\bar{\Omega}_0)_{4p-4} = F(x^{(\omega)} + S_0)$ where $\omega = \sum_k (p-1)\varepsilon_k$. But, by 4.2(1) and its proof, this space generates $\bar{\Omega}_0$ as K-module which leads to the contradiction $N = \bar{\Omega}_0$. Therefore, $N_{4p-4} = 0$. Now N has length at most $4p-6$ and is hence, by 1.1, isomorphic to $L(\lambda)$ with λ an exceptional weight other than $(0,0)$ (see 4.4). Since $\lambda \in \{(0,-4),(-1,-1),(-1,-3)\}$ and $x_3.m = -5m$ for any $m \in N_{4p-5}$ (see 4.1(5)), it follows that $N_{4p-5} = 0$. In particular, N has length at most $4p-7$ so that, by 3.2, $p = 3$ and $N_{4p-6} = N_6 \neq 0$. Since $x_3.m = 6m = 0$ for $m \in N_6$, this says $N_6 \cong L_0(-1,-3)$ as K_0-modules. A contradiction is obtained by observing that $(\Omega_0)_6$ (and hence N_6) contains no nonzero vector of weight $(-1,-3)$. Hence $\bar{\Omega}_0$ is simple, as desired.

(2) As in part (1), it is enough to prove that $\bar{\Omega}_1$ is simple as a graded K-module. Let N be a simple proper graded submodule of $\bar{\Omega}_1$. Here, $(\bar{\Omega}_1)_i = 0$ if either $i < 3$ or $i > 4p-3$ (see 4.3). Assume $N_{4p-3} \neq 0$. Then $N_{4p-3} = (\bar{\Omega}_1)_{4p-3}$ since this latter space is isomorphic to the simple K_0-module $L_0(-1,-3)$. By 4.2(2) and its proof, $(\bar{\Omega}_1)_{4p-3}$ generates $\bar{\Omega}_1$ as K-module giving the contradiction $N = \bar{\Omega}_1$. Hence $N_{4p-3} = 0$. Therefore, N has length at most $4p-7$ which implies $p = 3$ and $N_{4p-4} = N_8 \neq 0$. Once again, by reason of length, $N \cong L(\lambda)$ with λ an exceptional weight other than $(0,0)$. In particular, $\lambda \in \{(0,-4),(-1,-1)\}$ since $x_3.m = 8m$ for $m \in N_8$. By part (1), $L(0,-4) \cong \bar{\Omega}_0$ which has length 7. Hence $\lambda = (-1,-1)$. As in the proof of part (1), a contradiction is now obtained by observing that N_8 contains no nonzero vector of weight $(-1,-1)$. Thus, $\bar{\Omega}_1$ is simple and the proof is complete. □

5. Contragredient modules and dimensions of simple modules.

Let L be a Lie algebra and let V be an L-module. The *contragredient* of V is the space $V^* := \mathrm{Hom}_F(V,F)$ on which L acts according to the rule $(x.\varphi)(v) = -\varphi(x.v)$ ($x \in L$, $\varphi \in V^*$, $v \in V$). If L is graded and V is a graded L-module, then V^* is also graded by setting $(V^*)_i = \{\varphi \in V^* \mid \varphi(V_j) = 0 \text{ for all } j \neq -i\}$. In particular, $(V^*)_i \neq 0$ if and only if $V_{-i} \neq 0$.

Recall the convention stated before that if $\lambda \in \Lambda$, then λ_i is viewed as an integer with $-p < \lambda_i \leq 0$ by identifying $\mathbb{F}_p$ with $\mathbb{Z}/p\mathbb{Z}$ and using the coset representative in the indicated range.

5.1 Lemma ([5, Lemma 2]). *For $\lambda \in \Lambda$, $\dim_F L_0(\lambda) = 1 - \lambda_1$.*

5.2 Lemma. *$L_0(\lambda)^* \cong L_0(\lambda_1, -\lambda_2)$ as K_0-modules.*

Proof. By 5.1, $\dim_F L_0(\lambda) = 1 - \lambda_1$ so it follows that $L_0(\lambda)^* \cong L_0(\lambda_1, c)$ for some $c \in \mathbb{F}_p$. Let $\varphi \in L_0(\lambda)^*$. For any $v \in L_0(\lambda)$, $(x_3.\varphi)(v) = -\varphi(x_3.v) = -\varphi(\lambda_2 v) = (-\lambda_2\varphi)(v)$ so that $x_3.\varphi = -\lambda_2\varphi$. This says $c = -\lambda_2$ which completes the proof. □

5.3 Theorem.

(1) *If $\lambda \in \Lambda$ is not exceptional, then $L(\lambda)^* \cong L(\lambda_1, -\lambda_2 - 4)$.*
(2) $L(0,0)^* \cong L(0,0)$.
(3) $L(0,-4)^* \cong L(-1,-1)$.
(4) $L(-1,-3)^* \cong L(-1,-3)$.

Proof. (1) Assume $\lambda \in \Lambda$ is not exceptional. Then 1.1 says $L(\lambda) \cong Z(\lambda)$. The homogeneous component of $Z(\lambda)$ of least degree is $T^\omega \otimes L_0(\lambda)$ where $\omega = \sum_k (p-1)\varepsilon_k$. This is a simple K_0-module and is therefore, using 5.1, isomorphic to $L_0(\lambda_1, c)$ for some $c \in \mathbb{F}_p$. By 3.1(3), x_3 acts on this space by the scalar $\lambda_2 + 4$. Hence $c = \lambda_2 + 4$. It follows from 5.2 that the homogeneous component of $Z(\lambda)^*$ of greatest degree is K_0-isomorphic to $L_0(\lambda_1, -\lambda_2 - 4)$. The result now follows from the remarks before 1.1.

(2) Again, using the remarks before 1.1, $L(0,0)$ is the one-dimensional trivial module, so the claim is obvious.

(3) Using 4.5 and 4.3 it is easy to see that the homogeneous component of $L(0,-4)$ of least degree is $L(0,-4)_1$ which is K_0-isomorphic to $L_0(-1,1)$. The result now follows from 5.2 as in the proof of part (1).

(4) Here, the homogeneous component of $L(-1,-3)$ of least degree is $L(-1,-3)_3$ which is K_0-isomorphic to $L_0(-1,3)$. (This part also follows from the previous parts using the process of elimination.) □

5.4 Lemma. $Z(\lambda)^* \cong Z(\lambda_1, -\lambda_2 - 4)$ *(*$\lambda \in \Lambda$*).*

Proof. If λ is not exceptional, the statement follows from 5.3 and 1.1.

Assume $\lambda = (0,-4)$. By 4.2(1) it is enough to prove $\Omega_0^* \cong Z(0,0)$. The homogeneous component of Ω_0 of least degree is $(\Omega_0)_0$ which is K_0-isomorphic to $L_0(0,0)$. Hence, the homogeneous component of Ω_0^* of greatest degree is $(\Omega_0^*)_0 \cong ((\Omega_0)_0)^* \cong L_0(0,0)$. Therefore, a K^+-isomorphism $L_0(0,0) \to (\Omega_0^*)_0$ induces a K-homomorphism $\varphi : Z(0,0) \to \Omega_0^*$ which sends $T^\omega \otimes L_0(0,0)$ ($\omega = \sum_k (p-1)\varepsilon_k$) onto $(\Omega_0^*)_{-4p+4} \neq 0$ (using 4.1). By 3.3, φ is injective. Since both spaces have dimension p^3, φ is an isomorphism.

Next, assume $\lambda = (-1,-3)$. By 4.2(2), it is enough to prove that $V^* \cong Z(-1,-1)$ where $V := \Omega_1 / \operatorname{im} \tau_0$. The homogeneous component of V of least degree is the two-dimensional space V_1 (see 2.4) which is K_0-isomorphic to $L_0(-1,1)$. This yields a K-homomorphism $\varphi : Z(-1,-1) \to V^*$ which sends $T^\omega \otimes L_0(-1,-1)$ onto $(V^*)_{-4p+3} \neq 0$. By 3.3, φ is injective and, since both spaces have dimension $2p^3$, φ is an isomorphism. □

5.5 Theorem.

(1) *If* $\lambda \in \Lambda$ *is not exceptional, then* $\dim_F L(\lambda) = p^3(1 - \lambda_1)$.
(2) $\dim_F L(0,0) = 1$.
(3) $\dim_F L(0,-4) = \dim_F L(-1,-1) = p^3 - 1$.
(4) $\dim_F L(-1,-3) = p^3 - 2$.

Proof. (1) Use 1.1 and 5.1.

(2) $L(0,0)$ is the trivial one-dimensional module.

(3) Use 4.5(1), 4.3 and 5.3(3).

(4) Use 4.5(2) and 4.3. □

6. Cartan invariants.

Let V be a K-module and let $[V]$ denote the element of the Grothendieck group of $u(K)$ corresponding to V. Then $[V]$ can be written uniquely in the form $[V] = \sum_{\lambda \in \Lambda} [V : L(\lambda)][L(\lambda)]$ with $[V : L(\lambda)] \in \mathbb{Z}$. In fact, $[V : L(\lambda)]$ is just the multiplicity of $L(\lambda)$ as a composition factor of V.

6.1 Lemma.

(1) *If* $\lambda \in \Lambda$ *is not exceptional, then* $[Z(\lambda)] = [L(\lambda)]$.
(2) $[Z(0,-4)] = [L(0,-4)] + [L(0,0)]$.
(3) $[Z(-1,-3)] = [L(-1,-3)] + 3[L(0,0)] + [L(0,-4)]$.

(4) $[Z(-1,-1)] = [L(-1,-1)] + 3[L(0,0)] + [L(-1,-3)]$.
(5) $[Z(0,0)] = [L(0,0)] + [L(-1,-1)]$.

Proof. (1) Use 1.1.

(2) By 4.2(1), $Z(0,-4) \cong \Omega_0$. According to 2.3(2), $L(0,0) \cong \ker\delta_0 < \Omega_0$ and, by 4.5(1), $\Omega_0/\ker\delta_0 = \bar{\Omega}_0 \cong L(0,-4)$.

(3) By 4.2(2), $Z(-1,-3) \cong \Omega_1/\operatorname{im}\tau_0$. Now $\operatorname{im}\delta_0 \leq \ker\delta_1 \leq \Omega_1$. Therefore, if $\pi : \Omega_1 \to \Omega_1/\operatorname{im}\tau_0$ denotes the canonical epimorphism, then $\pi(\operatorname{im}\delta_0) \leq \pi(\ker\delta_1) \leq \pi(\Omega_1)$. As in the proof of 4.3, $\operatorname{im}\tau_0 \cap \ker\delta_1 = \{0\}$, so $\pi(\operatorname{im}\delta_0) \cong \operatorname{im}\delta_0 \cong \bar{\Omega}_0 \cong L(0,-4)$ (by 4.5(1)), $\pi(\ker\delta_1)/\pi(\operatorname{im}\delta_0) \cong \ker\delta_1/\operatorname{im}\delta_0$ which has three composition factors, each isomorphic to $L(0,0)$ (by 2.3(2)), and $\pi(\Omega_1)/\pi(\ker\delta_1) \cong \bar{\Omega}_1 \cong L(-1,-3)$ (by 4.5(2)).

(4) Use 5.4, 5.3 and part (3).

(5) Use 5.4, 5.3 and part (2). □

The Lie algebra K_0 has a p-grading: $K_0 = (K_0)_{-1}\dot{+}(K_0)_0\dot{+}(K_0)_1$ where $(K_0)_{-1} = Fx^{(2\varepsilon_2)}$, $(K_0)_0 = Fx^{(\varepsilon_1+\varepsilon_2)} + Fx_3$ and $(K_0)_1 = Fx^{(2\varepsilon_1)}$. The component $(K_0)_0 =: T$ is a maximal torus for K. Also, $(K_0)_1 \lhd T + (K_0)_1 := K_0^+$ and $K_0^+/(K_0)_1 \cong T$ so that any T-module becomes a K_0^+-module in the natural way.

For $\lambda \in \Lambda$, let F_λ denote the one-dimensional T-module on which $x^{(\varepsilon_1+\varepsilon_2)}$ (resp., x_3) acts as multiplication by λ_1 (resp., λ_2). Then $Z_0(\lambda) := u(K_0) \otimes_{u(K_0^+)} F_\lambda$ has unique simple quotient $L_0(\lambda)$.

The following lemma gives the composition factors of $Z_0(\lambda)$ (by way of formulas in the Grothendieck group for $u(K_0)$). It follows easily from [6, Theorem 1] and [2, Theorem 5.1].

6.2 Lemma. $[Z_0(\lambda)] = \begin{cases} [L_0(\lambda)] + [L_0(2-p-\lambda_1,\lambda_2)] & \lambda_1 \neq 1-p, \\ [L_0(\lambda)] & \lambda_1 = 1-p. \end{cases}$

In the following discussion, all parametrizations by Λ are assumed to be relative to a fixed ordering.

Let X be the $p^2 \times 1$-matrix with λ-entry

$$X_\lambda = \sum_{\mu,\nu} [Z_0(\nu) : L_0(\mu)][Z(\mu) : L(\lambda)].$$

6.3 Lemma. $X_\lambda = \begin{cases} 16 & \lambda = (0,0), \\ 4 & \lambda = (0,-4), (-1,-1) \text{ or } (-1,-3), \\ 1 & \lambda_1 = 1-p, \\ 2 & \text{otherwise.} \end{cases}$

Proof. Assume $\lambda_1 \neq 1-p$. Then 6.1 implies $[Z(1-p,c) : L(\lambda)] = 0$ for all $c \in \mathbb{F}_p$. Hence, $X_\lambda = \sum_{\mu \neq (1-p,c)} 2[Z(\mu) : L(\lambda)]$ by 6.2. Using 6.1 the result follows.

Now assume $\lambda_1 = 1-p$. If $\mu_1 \neq 1-p$, then $[Z(\mu) : L(\lambda)] = 0$. Therefore, $X_\lambda = \sum_{\mu_1=1-p}[Z(\mu) : L(\lambda)] = 1$, again by 6.2 and 6.1. □

Let C be the Cartan matrix of K. Hence, C is the $p^2 \times p^2$-matrix with (λ,μ)-entry $C_{\lambda\mu} := [P(\lambda) : L(\mu)]$ where $P(\lambda)$ is the projective cover of $L(\lambda)$.

6.4 Theorem. *Assume $p > 3$. $C = p^\beta X\, {}^tX$ where $\beta = p^3 - 12$ and ${}^tX$ denotes the transpose of X.*

Proof. Note that for $\lambda \in \Lambda$, ${F_\lambda}^* \cong F_{-\lambda}$. According to [3], $C = p^\beta\, {}^tR'\, {}^tQ'UQR$ where

$$\begin{aligned} Q_{\lambda\mu} &= [Z_0(\lambda) : L_0(\mu)], \\ Q'_{\lambda\mu} &= [Z_0(-\lambda) : L_0(\mu)], \\ R_{\lambda\mu} &= [Z(\lambda) : L(\mu)], \\ R'_{\lambda\mu} &= [Z(\lambda) : L(\mu)^*], \\ U_{\lambda\mu} &= 1. \end{aligned}$$

Evidently, $R' = RP_1$ and $Q' = P_2Q$ for some permutation matrices P_1 and P_2. Since ${}^tP_2U = U$, it follows that $C = p^\beta\, {}^tP_1\, {}^tR\, {}^tQUQR = p^\beta\, {}^tP_1X\, {}^tX$. By 5.3, P_1 is obtained from the identity matrix by transposing the $(0,-4)$- and $(-1,-1)$-columns and transposing the (λ_1,λ_2)- and $(\lambda_1,-\lambda_2-4)$-columns for λ not exceptional. Hence, 6.3 implies ${}^tP_1X = X$. □

6.5 Theorem. *Assume $p > 3$. For $\lambda \in \Lambda$, $\dim_F P(\lambda) = p^\alpha X_\lambda$ where $\alpha = p^3 - 6$ and X_λ is as in 6.4.*

Proof. $\dim_F Z(\mu) = p^3 \dim_F L_0(\mu)$ and $\dim_F Z_0(\nu) = p$, so that

$$\begin{aligned} \sum_\lambda X_\lambda \dim_F L(\lambda) &= \sum_{\nu,\mu,\lambda} [Z_0(\nu) : L_0(\mu)][Z(\mu) : L(\lambda)] \dim_F L(\lambda) \\ &= p^3 \sum_{\nu,\mu} [Z_0(\nu) : L_0(\mu)] \dim_F L_0(\mu) \\ &= p^4 \sum_\nu 1 = p^6. \end{aligned}$$

Therefore, denoting by $\dim P$ (resp., $\dim L$) the $p^2 \times 1$-matrix with λ-entry $\dim_F P(\lambda)$ (resp., $\dim_F L(\lambda)$), 6.4 implies $\dim P = C \dim L = p^\beta X\, {}^tX \dim L = p^\alpha X$. □

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